J. Differential Equations 251 (2011) 2685–2695

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A fast–slow dynamical systems theory for the Kuramoto type phase model

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ARTICLE INFO

Article history:
Received 14 January 2011
Revised 27 March 2011
Available online 22 April 2011

Keywords:
A fast–slow system
Kuramoto model
Synchronization
Order parameter

ABSTRACT

We present a fast–slow dynamical systems theory for the Kuramoto type phase model. When the order parameters are frozen, the fast system consists of independent oscillator equations, whereas the slow system describes the evolution of order parameters. We average out the slow system over the fast manifold to derive a weak form of an amplitude–angle coupled system for the evolution of Kuramoto's order parameters. This yields the slow evolution of order parameters to be constant values which gives a rigorous proof to Kuramoto's original assumption in his self-consistent mean-field theory.

1. Introduction

The purpose of this paper is to present a fast–slow dynamical systems theory for the Kuramoto type phase model without relying on *a priori* assumptions commonly used in statistical physics and nonlinear dynamics for synchronization. The mathematical treatment for the synchronized phenomena was pioneered by Winfree and Kuramoto in [8,9,20]. They introduced phase models for large weakly coupled oscillator system and showed that synchronized behavior of complex biological systems can emerge from the competing mechanisms of intrinsic randomness and nonlinear attractive couplings. Kuramoto oscillators can be visualized as point active rotors moving on the unit circle $\mathbb{S}^1$.

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1 The work of S.-Y. Ha was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2009-0093137) and research grant of SNU-CNS.
2 The work of M. Slemrod was supported by National Institute of Mathematical Sciences of Korea (2010 Thematic Program "Kinetic equations and Plasma physics").

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.04.004

More precisely, let x_i be the position of the i-th rotor and we write $x_i = e^{i\theta_i}$. In this situation, the dynamics of Kuramoto oscillators are governed by the following phase model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t \geqslant 0,\ i = 1, \ldots, N, \tag{1}$$

subject to initial data:

$$\theta_i(0) = \theta_{i0}, \tag{2}$$

where K and Ω_i are the uniform positive coupling strength, and the intrinsic natural frequency of the i-th oscillator drawn from some distribution function $g = g(\Omega)$ respectively. The explicit form of g is irrelevant in the following analysis. Mathematical results up to 2005 for the Kuramoto model (1) can be found in the survey papers [1,5,15,16].

The standard approach [8,9] for the synchronization initiated by Kuramoto is to employ the complex order parameter to measure the degree of synchronization:

$$r(t)e^{i\phi(t)} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \quad t \geqslant 0. \tag{3}$$

Note that the real order parameters r and ϕ are functions of all phases θ_i, hence they are dependent on K and N implicitly, i.e., $r = r(K, t, N),\ \phi = \phi(K, t, N)$.

The novelty of this paper is to give a rigorous mathematical underpinning for Kuramoto's original guess of constant order parameters r and ϕ in his self-consistent theory [9,10]. We do this via a fast–slow dynamics decomposition for the finite-dimensional Kuramoto model. Our rationale deriving the fast–slow decomposition for such a Kuramoto type system is based on the idea that the dynamics of individual oscillators are rapidly oscillating, whereas the Kuramoto order parameters r, ϕ are slowly evolving. The fast dynamics consists of N-individual oscillator equations coupled with other oscillators only through the order parameters, while the slow dynamics comprised of two scalar coupled ordinary differential equations for r and ϕ. In fact, Kuramoto guessed [8,9] that long term behavior would yield constant r and ϕ in thermodynamic limit, and hence the oscillator system is asymptotically governed by the decoupled oscillator equations with supplemented by constants r and ϕ. In fact, to be consistent with his ansatz for constant r, Kuramoto argued that the contribution from the drifting oscillators in the order parameter r in thermodynamic limit is asymptotically negligible. But as noted by Strogatz [16] in his 2000 survey:

"*In the last of her three Bowen lectures at Berkeley in* 1986, *Kopell pointed out that Kuramoto's argument contained a few intuitive leaps that were far from obvious. In fact, they began to seem paradoxical the more one thought about them, and she wondered whether one could prove some theorems that would put the analysis on firmer footing. In particular, she wanted to redo the analysis rigorously for large but finite N, and then prove a convergence result as* $N \to \infty$. *But it would not be easy. Whereas Kuramoto's approach had relied on the assumption that r was strictly constant, Kopell emphasized that nothing like that could be strictly true for any finite N. Think about the simple case* $K = 0$. *Then* $\dot{\theta}_i = \Omega_i$ *and every trajectory is dense on the N-torus, at least for the generic case where the frequencies are rationally independent. But then* $r(t)$ *eventually passes through every possible value between* 0 *and* 1, *completely unlike the constant value* $r = 0$ *implied by Kuramoto's argument! Admittedly,* $r(t)$ *would spend nearly all its time very close to zero, at* $r = \mathcal{O}(N^{-\frac{1}{2}}) \ll 1$, *and only blip up extremely rarely—in that sense* $r \equiv 0$ *is practically correct. But how can this rough idea be made precise? When* $K \neq 0$, *the situation would become still more difficult, because now there would be three subpopulations of oscillators—locked and drifting ones as in Kuramoto's analysis, but also some fuzzy oscillators between them, determined by the ever-fluctuating boundary* $\Omega_i \approx Kr(t)$."

In this paper, by direct application of Artstein–Kevrekidis–Slemrod–Titi's unified theory (in short AKST's theory) for singular perturbation, we prove that the slow motion is just r and ϕ constant, and we thus validate Kuramoto's 1975 assumption in [8].

This paper consists of three sections. In Section 2, we review AKST's unified theory. In Section 3, we revisit Kuramoto's order parameter approach via AKST's theory. In particular we derive a fast–slow dynamical systems theory and apply the Young measure approach to obtain the evolution of slow dynamical variables and in fact we prove that they are constant.

2. Preliminaries

In this section, we review invariant measures, Young measures and Artstein–Kevrekidis–Slemrod–Titi's unified approach to singular perturbations [11,13,14,18].

2.1. Invariant measures and Young measures

In this part, we collect some basic notions from [2] on the invariant measures and Young measures to be crucially used in later sections. For detailed discussions, we refer to [4,12,19].

Recall that a probability measure μ on $\mathbb{R}^N$ is a σ-additive set function defined on the Borel subsets of $\mathbb{R}^N$ with values in $[0,1]$ and $\mu(\mathbb{R}^N)=1$. We set $\mathcal{P}(\mathbb{R}^N)$ to be the family of all probability measures on $\mathbb{R}^N$ endowed with weak convergence of measures.

Definition 2.1. Let μ be a probability measure defined on $\mathbb{R}^N$.

1. The support of μ (often denoted by $\mathrm{spt}(\mu)$) is the smallest closed set C such that $\mu(C)=1$.
2. μ is an invariant measure of the system

$$\frac{dx}{dt}=f(x), \quad f\colon \text{Lipschitz continuous}, \tag{4}$$

   if the solutions $X(t,x_0)$ to (4) for x_0 in a neighborhood of $\mathrm{spt}(\mu)$ are defined on a common interval I around $t=0$ and if $\mu(B)=\mu(X(t,B))$ for each $t\in I$ and every Borel set B.
3. μ is a Young measure if $\mu(\cdot):[a,b]\to\mathcal{P}(\mathbb{R}^N)$ is a measurable map.

We also recall the definition of convergence of measures as follows.

Definition 2.2. (See [6].) Let (μ_j) be a sequence of Young measures defined on the same interval $[a,b]$. The sequence μ_j converges to the Young measure μ_0 if and only if

$$\int_a^b\int_{\mathbb{R}^N} h(x,t)\,\mu_j(t)(dx)\,dt \to \int_a^b\int_{\mathbb{R}^N} h(x,t)\,\mu_0(t)(dx)\,dt,$$

for every continuous and bounded real function $h=h(x,t)$.

Remark 2.1. 1. The continuity of a test function h in time-variable can be replaced by measurability.

2. Usual point-valued function $x=x(\cdot)$ can be viewed as a Young measure, when the point $x(t)$ is identified with the Dirac measure supported on the singleton $\{x(t)\}$. Hence when we refer to the convergence of a sequence of functions in the sense of Young measures, we mean the convergence in the sense of Definition 2.2 for the corresponding Dirac measure-valued maps. Thus when we have a sequence of continuous functions uniformly bounded in j, $\{x_j(t)\}$, $a\leqslant t\leqslant b$, its associated Young

measures are $\mu_j(t) = \delta(x - x_j(t))$. If we choose $h(x,t) = a(x)b(t)$, then convergence of $x_j(\cdot)$ to the Young measure μ_0 in Young measures implies the statement

$$\int_a^b b(t)a\big(x_j(t)\big)\,dt \to \int_a^b b(t)\bigg(\int_{\mathbb{R}^N} a(x)\,\mu_0(t)(dx)\bigg)\,dt.$$

Hence the weak-$*$ $L_\infty([a,b])$ limit of the functions $a(x_j(\cdot))$ is represented by the value

$$\int_{\mathbb{R}^N} a(x)\,\mu_0(t)(dx).$$

This representation has proved extremely valuable in applications [17].

2.2. Review on AKST's unified approach

Consider a fast–slow system:

$$\frac{dU}{dt} = \frac{F(U)}{\varepsilon} + G(U), \quad U \in \mathbb{R}^N,\ t > 0, \tag{5}$$

subject to initial data:

$$U(0) = U_0, \qquad t = 0, \tag{6}$$

where $F(U), G(U) : \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions and denote fast and slow parts of the system (5) respectively.

We first present a framework of AKST's theory:

- ($\mathcal{F}1$) The solutions of (5) lie in a compact set $H \subset \mathbb{R}^N$ on some interval, say $0 \leqslant t \leqslant 1$ for any $0 < \varepsilon \ll 1$, and in addition, there is a compact set $K \subset H$ which is positively invariant with respect to the fast part of (5):

$$\frac{dU}{ds} = F(U). \tag{7}$$

- ($\mathcal{F}2$) For initial data $U_0 \in K$ solutions of the full system (5) and fast system (7) are unique.

Theorem 2.1. *Let U^ε be solutions of* (5) *satisfying $U_0^\varepsilon = U_0 \in K$, and defined on a common interval, say* $[0,1]$. *Then for every sequence $\varepsilon_j \to 0$, there exists a subsequence $U^{\varepsilon_j}(\cdot)$ which converges in the sense of Young measures to a Young measure, say $\mu_0(\cdot)$ defined on* $[0,1]$. *The value of the limit Young measure is an invariant measure for the fast equation* (7)*.*

Definition 2.3. Let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous function called as a measurement.

(i) For a given probability measure μ, we call the action of V on a measure μ:

$$\hat{V}(\mu) := \int_{\mathbb{R}^N} V(\lambda)\,\mu(d\lambda)$$

as an observable.

(ii) The observable $\hat{V}(\mu)$ is an "*orthogonal observable*" of the fast part to (5), if the measurement $V(\cdot)$ is a first integral of the fast system (7), i.e., $V(U(s))$ is constant along any solution of (7) and hence is equivalent to the relation $\nabla_U V \cdot F \equiv 0$ if V is differentiable. Here $a \cdot b$ is the standard Euclidean inner product of two vectors $a, b \in \mathbb{R}^N$.

Theorem 2.2. *Suppose the assumptions* $(\mathcal{F}1)$–$(\mathcal{F}2)$ *hold, and let* $U^{\varepsilon_j}(\cdot)$ *be the solutions of* (5) *satisfying* $U^{\varepsilon_j}(0) = U_0$ *and defined on, say* $[0,1]$, *and which converge to* $\mu_0(\cdot)$ *in the sense of Young measures via Theorem* 2.1. *Then for any orthogonal observable* $\hat{V}(\cdot)$ *of the system* (5), *the measurement* $V(U^{\varepsilon_j}(t))$ *converges in weak-*$*$ $L_\infty([0,1])$ *to* $\hat{V}(\mu_0(t))$:

$$\hat{V}\big(\mu_0(t)\big) = \int\limits_{\mathbb{R}^N} V(\lambda)\,\mu_0(t)(d\lambda).$$

Moreover, $\hat{V}(\mu_0(t))$ *satisfies the relation*:

$$\hat{V}\big(\mu_0(t)\big) = V(U_0) + \int\limits_0^t \int\limits_{\mathbb{R}^N} \nabla V(\lambda) \cdot G(\lambda)\,\mu_0(s)(d\lambda)\,ds. \tag{8}$$

Remark 2.2. One may ask why we do not differentiate the above integral relation (8) to obtain an ordinary differential equation for $\hat{V}(\mu_0(t))$. First we note even if we could differentiate the integral relation, it would not yield an ordinary differential equation in the classical sense, i.e., since the Young measure $\mu_0(t)$ is determined via the initial data U_0, the right-hand side depends on the initial data. Secondly the issue of differentiability has been covered in Theorem 6.5 of [2]. The sufficient conditions given there are that the Young measure μ_0 is uniquely determined by the initial data U_0 and furthermore, that it is Lipschitz continuous as a function of the data U_0. In our system, the continuity of the measure μ_0 as a function of the data is not expected, but fortunately it will not be needed.

3. A revisit to Kuramoto's mean-field approach

In this section, we present a fast–slow dynamical systems theory for the Kuramoto model in thermodynamic limit (mean-field limit).

3.1. A fast–slow dynamics decomposition

In this part, we present a fast–slow dynamics decomposition for the Kuramoto model in the thermodynamic limit.

Consider a Kuramoto type system of ordinary differential equations

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^N a_j \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N,\ t > 0, \tag{9}$$

where a_j is a nonnegative constant. We next introduce the weighted Kuramoto order parameter $(r, \phi) \in \mathbb{R}_+ \times \mathbb{R}$:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^N a_j e^{i\theta_j}. \tag{10}$$

Note that r is always bounded, i.e., $0 \leqslant r \leqslant 1$.

We next divide (10) by $e^{i\theta_i}$ to get the equation:

$$re^{i(\phi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} a_j e^{i(\theta_j-\theta_i)},$$

and compare real and imaginary parts of the above relation to find

$$\begin{aligned} r\cos(\phi-\theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \cos(\theta_j-\theta_i), \\ r\sin(\phi-\theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j-\theta_i). \end{aligned} \tag{11}$$

Then it follows from (11) that the system (9) becomes

$$\dot{\theta}_i = \Omega_i + Kr\sin(\phi-\theta_i), \quad i=1,\dots,N,\ t>0. \tag{12}$$

Note that the system (12) looks decoupled, but the order parameters r and ϕ are functions of θ_j, $j=1,\dots,N$, hence in fact (12) corresponds to the rewriting of the original system (9). However as we can see from (12), the effect of neighboring oscillators to the dynamics of a given oscillator is only through the order parameters r and ϕ, and hence when the order parameters r and ϕ are constant, the dynamics of each oscillator is solvable in exact form.

Furthermore we differentiate Eq. (10) with respect to t to get

$$\dot{r}e^{i\phi} + ire^{i\phi}\dot{\phi} = \frac{i}{N}\sum_{j=1}^{N} a_j e^{i\theta_j}\dot{\theta}_j.$$

We divide the resulting equation by $e^{i\phi}$ to find

$$\dot{r} + ir\dot{\phi} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j-\phi)\dot{\theta}_j + \frac{i}{N}\sum_{j=1}^{N} a_j \cos(\theta_j-\phi)\dot{\theta}_j. \tag{13}$$

We now take real and imaginary parts of (13) to obtain

$$\begin{aligned} \dot{r} &= -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j-\phi)\dot{\theta}_j, \\ \dot{\phi} &= \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j-\phi)\dot{\theta}_j. \end{aligned} \tag{14}$$

Thus we combine (12) and (14) to get the evolutionary system:

$$\begin{aligned} \dot{\theta}_i &= \Omega_i - Kr\sin(\theta_i-\phi), \quad i=1,\dots,N,\ t>0, \\ \dot{r} &= -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j-\phi)\big(\Omega_j - Kr\sin(\theta_j-\phi)\big), \end{aligned}$$

$$\dot{\phi} = \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\big(\Omega_j - Kr\sin(\theta_j - \phi)\big). \tag{15}$$

We wish to study long-time dynamics and mean-field limit simultaneously and hence introduce the fast time $t = \frac{\tau}{\varepsilon}$ where $0 \leqslant \tau \leqslant 1$. Then the system becomes

$$\begin{aligned}
\varepsilon\dot{\theta}_i &= \Omega_i - Kr\sin(\theta_i - \phi), \quad i = 1, \ldots, N,\ t > 0,\\
\dot{r} &= -\frac{1}{\varepsilon N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\big(\Omega_j - Kr\sin(\theta_j - \phi)\big),\\
\dot{\phi} &= \frac{1}{r\varepsilon N}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\big(\Omega_j - Kr\sin(\theta_j - \phi)\big),
\end{aligned} \tag{16}$$

where dot derivative now denotes $\frac{d}{d\tau}$. Next we do the following steps. We set

$$\varepsilon N = 1$$

so that in principle as $\varepsilon \to 0$, we will get an infinite set of equations. However to preclude this event and keep our system finite dimensional, we set

$$a_j = \begin{cases} 1, & j \leqslant M,\\ 0, & j > M. \end{cases}$$

Now we have

$$\varepsilon\dot{\theta}_i = \Omega_i - Kr\sin(\theta_i - \phi), \quad i = 1, \ldots, M,\ 0 < \tau < 1, \tag{17}$$

$$\dot{r} = -\sum_{j=1}^{M} \sin(\theta_j - \phi)\big(\Omega_j - Kr\sin(\theta_j - \phi)\big), \tag{18}$$

$$\dot{\phi} = \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\big(\Omega_j - Kr\sin(\theta_j - \phi)\big). \tag{19}$$

This is a coupled fast (17) and slow (18)–(19) system. Furthermore since θ_j only enters the right-hand side of (17)–(19) through sin and cos, without loss of generality we may restrict θ_j to the interval $[-\pi, \pi]$ and identify all θ_i mod 2π as the same θ_i. Thus the fast–slow theory of Artstein–Vigodner [3] or more recently, Artstein, Kevrekidis, Slemrod and Titi [2] will apply to produce equations for the evolution of the phase variables (r, ϕ) as $\varepsilon \to 0$. Note that when slow variables r and ϕ are frozen in the first equation (17), Eq. (17) is in fact explicitly solvable. (See Appendix C in [7] for the derivation of analytic formula.) Below we just display the formula in the original t-variable $\tau = \varepsilon t$:

- Case 1 $(Kr > |\Omega_i|)$:

$$\begin{aligned}
t\sqrt{(Kr)^2 - \Omega_i^2} &= \log\left|\frac{\Omega_i \tan\frac{\theta_i(t)-\phi}{2} - Kr - \sqrt{(Kr)^2 - \Omega_i^2}}{\Omega_i \tan\frac{\theta_i(t)-\phi}{2} - Kr + \sqrt{(Kr)^2 - \Omega_i^2}}\right|\\
&\quad - \log\left|\frac{\Omega_i \tan\frac{\theta_{i0}-\phi}{2} - Kr - \sqrt{(Kr)^2 - \Omega_i^2}}{\Omega_i \tan\frac{\theta_{i0}-\phi}{2} - Kr + \sqrt{(Kr)^2 - \Omega_i^2}}\right|.
\end{aligned}$$

- Case 2 ($Kr = |\Omega_i|$):

$$t = \frac{2}{\Omega_i \tan\frac{\theta_{i0}-\phi}{2} - Kr} - \frac{2}{\Omega_i \tan\frac{\theta_i(t)-\phi}{2} - Kr}.$$

- Case 3 ($Kr < |\Omega_i|$):

$$\tan\frac{\theta_i(t)-\phi}{2} = \frac{1}{R_i^\infty}\left\{\sqrt{\left(R_i^\infty\right)^2 - 1}\tan\left[\frac{Krt}{2}\sqrt{\left(R_i^\infty\right)^2 - 1} + \tan^{-1}\left(\frac{R_i^\infty \tan\frac{\theta_{i0}}{2} - 1}{\sqrt{(R_i^\infty)^2 - 1}}\right)\right] + 1\right\}, \tag{20}$$

where R_i^∞ is defined by the relation $R_i^\infty = \frac{\Omega_i}{Kr}$.

3.2. Application of Young measure approach

In this part, we present an application of AKST theory to the fast–slow system (17)–(19).
If we define

$$U := (\theta, r, \phi) \in \mathbb{R}^{M+2}, \qquad F(U) := \left(\Omega_i - Kr\sin(\theta_i - \phi), 0, 0\right),$$

$$G(U) := \left(0, -\sum_{j=1}^{M}\sin(\theta_j - \phi)\left(\Omega_j - Kr\sin(\theta_j - \phi)\right), \frac{1}{r}\sum_{j=1}^{M}\cos(\theta_j - \phi)\left(\Omega_j - Kr\sin(\theta_j - \phi)\right)\right),$$

the system (17)–(19) can be written in compact form:

$$\dot{U} = \frac{F(U)}{\varepsilon} + G(U), \tag{21}$$

which is the fast–slow formulation of Artstein, Kevrekidis, Slemrod and Titi [2]. The assumption of ($\mathcal{F}1$) requires that the unique solution of

$$\dot{U} = F(U), \qquad U(0) = U_0 \tag{22}$$

lies in a compact subset of $\mathbb{R}^{M+2}$. This is trivially satisfied, since the slow components (r, ϕ) satisfy

$$\dot{r} = 0, \qquad \dot{\phi} = 0,$$

and the fast components θ lie in the compact M-torus $\mathbb{T}^M$. The theory assumes ($\mathcal{F}1$)–($\mathcal{F}2$) and the full system (21) has unique solutions on some finite interval, say $0 \leqslant \tau \leqslant 1$. This is certainly true for initial data in $\mathbf{H}(\delta)$:

$$\mathbf{H}(\delta) := \left\{(r, \phi, \theta)\colon\ 0 < \delta \leqslant r,\ \phi \in \mathbb{T}^1,\ \theta \in \mathbb{T}^M\right\}.$$

The next step is to identify an *orthogonal* measurement $V(U)$ for which

$$\nabla V(U) \cdot F(U) = 0,$$

which in our case means

$$\sum_{i=1}^{M}\frac{\partial V}{\partial \theta_i}\left(\Omega_i - Kr\sin(\theta_i - \phi)\right) = 0.$$

It is easy to see that any functions of r and ϕ will suffice as a measurement. In particular, we choose $V(\lambda_1, \dots, \lambda_{M+2}) = \lambda_{M+1}, \lambda_{M+2}$, i.e., $V = r$, $V = \phi$ as our two measurements which of course yield orthogonal observables. By Theorem 2.1, solutions of (21) for initial data $(r, \phi, \theta) \in \mathbf{H}(\delta)$, defined on $0 \leqslant \tau \leqslant 1$ will have a convergent subsequence $U^{\varepsilon_j}(\cdot)$ which converges in the sense of Young measures to a Young measure $\mu_0(\cdot)$ defined on $[0, 1]$. The values of the limit Young measure are invariant measures of the fast system (22). Since for the fast system (22), oscillators are decoupled, hence $\mu_0(t)$ is a product measure:

$$\mu_0(\tau)(d\lambda) = \nu_1(\tau)(d\lambda_1) \otimes \cdots \otimes \nu_M(\tau)(d\lambda_M) \otimes \nu_{M+1}(d\lambda_{M+1}) \otimes \nu_{M+2}(d\lambda_{M+2}), \tag{23}$$

and each $\nu_i(t)(d\lambda_i)$, $1 \leqslant i \leqslant M$, is itself an invariant probability measure for the i-th oscillator equation of (17). Since $\frac{dr}{d\tau} = 0$, $\frac{d\phi}{d\tau} = 0$ in the fast system,

$$\nu_{M+1}(\tau) = \delta\big(\lambda_{M+1} - r(\tau)\big), \qquad \nu_{M+2}(\tau) = \delta\big(\lambda_{M+2} - \phi(\tau)\big).$$

Furthermore by Theorem 2.2, we use the orthogonal observables associated with any measurement V satisfying the integral equation:

$$\hat{V}_0\big(\mu_0(\tau)\big) = V(U_0) + \int_0^\tau \int_{\mathbb{R}^{M+2}} \nabla V(\lambda) \cdot G(\lambda)\, \mu_0(s)(d\lambda)\, ds. \tag{24}$$

In our example the choices $V = r$, $V = \phi$ yield the limit slow evolution for r and ϕ:

$$\begin{aligned} r(\tau) &= r(0) - \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^M} \sin(\lambda_j - \phi)\big(\Omega_j - Kr\sin(\lambda_j - \phi)\big)\, \mu_0(s)(d\lambda_1 \cdots d\lambda_M)\, ds, \\ \phi(\tau) &= \phi(0) + \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^M} \frac{1}{r}\cos(\lambda_j - \phi)\big(\Omega_j - Kr\sin(\lambda_j - \phi)\big)\, \mu_0(s)(d\lambda_1 \cdots d\lambda_M)\, ds. \end{aligned} \tag{25}$$

We now substitute (23) into (25) to obtain

$$\begin{aligned} r(\tau) &= r(0) - \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}} \sin(\lambda_j - \phi)\big(\Omega_j - Kr\sin(\lambda_j - \phi)\big)\, \nu_j(s)(d\lambda_j)\, ds, \\ \phi(\tau) &= \phi(0) + \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}} \frac{1}{r}\cos(\lambda_j - \phi)\big(\Omega_j - Kr\sin(\lambda_j - \phi)\big)\, \nu_j(s)(d\lambda_j)\, ds. \end{aligned} \tag{26}$$

System (26) produces what is usually called an amplitude equation for r, but in fact it is the coupled system (26) that determines r.

We can say more about $\mu_0(\tau)$ when r lies in the domain

$$-1 < \frac{\Omega_i}{Kr} < 1, \quad \text{for some } i,\ 1 \leqslant i \leqslant M. \tag{27}$$

There exist two values θ_{ik}^e, $k = 1, 2$, satisfying

$$\theta_{ik}^e \equiv \phi(0) + \sin^{-1}\left(\frac{\Omega_i}{Kr}\right), \qquad \big|\theta_{i1}^e - \phi(0)\big| < \frac{\pi}{2}, \qquad \big|\theta_{i2}^e - \phi(0)\big| > \frac{\pi}{2}.$$

For initial data $\theta_i(0) \neq \theta_{i2}^e$, all solutions converge to the asymptotically stable equilibrium θ_{i1}^e as $t \to \infty$. Hence the limit measure $\mu_0(\tau)$ in (23) must be of the form

$$\nu_i(\tau) = \delta\big(\lambda_i - \theta_{i1}^e\big) \quad \text{or} \quad \nu_i(\tau) = \delta\big(\lambda_i - \theta_{i2}^e\big),$$

where of course the choice depends respectively on whether the initial data is at θ_{i2}^e or not. In Case 2, the two equilibrium points of Case 1 coalesce and

$$\nu_i(d\lambda_i) = \delta\big(\lambda_i - \theta_{i3}^e\big), \quad \text{where } \frac{\theta_{i3}^e - \phi}{2} = \tan^{-1}\left(\frac{Kr}{\Omega_i}\right).$$

In Case 3, it follows from (20) that $\theta_i(t)$ is a periodic function with a minimal period T_i in t:

$$T_i := \left[\frac{Kr}{2}\sqrt{\big(R_i^\infty\big)^2 - 1}\right]^{-1}\pi.$$

In this case, the invariant measure is supported on the inverse image of periodic solution:

$$\nu_i(d\lambda) = \frac{1}{T_i}\, d\theta_i^{-1}(\lambda_i), \quad \text{where } \theta_i^{-1} \text{ denotes the inverse of } \theta_i.$$

Hence we can easily compute

$$\begin{aligned}
&\int_{\mathbb{R}} \sin(\lambda_i - \phi)\big(\Omega_i - Kr\sin(\lambda_i - \phi)\big)\,\nu_i(s)(d\lambda_i) \\
&\quad = \frac{1}{T_i}\int_{\mathbb{R}} \sin(\lambda_i - \phi)\big(\Omega_i - Kr\sin(\lambda_i - \phi)\big)\,d\theta_i^{-1}(\lambda_i) \\
&\quad = \frac{1}{T_i}\int_0^{T_i} \sin\big(\theta_i(s_i) - \phi\big)\big(\Omega_i - Kr\sin\big(\theta_i(s_i) - \phi\big)\big)\,ds_i \quad \text{by } s_i := \theta_i^{-1}(\lambda_i) \\
&\quad = \frac{1}{T_i}\int_0^{T_i} \sin\big(\theta_i(s_i) - \phi\big)\dot{\theta}_i(s_i)\,ds_i \\
&\quad = -\frac{1}{T_i}\int_0^{T_i} \left(\frac{d}{ds_i}\cos\big(\theta_i(s_i) - \phi\big)\right) ds_i \\
&\quad = 0, \quad \text{by the periodicity of } \theta_i.
\end{aligned}$$

Similarly we have

$$\int_{\mathbb{R}} \cos(\lambda_j - \phi)\big(\Omega_j - Kr\sin(\lambda_j - \phi)\big)\,\nu_j(s)(d\lambda_j)\,ds = 0.$$

Thus no matter what value $r(\tau)$ takes in the slow motion, (26) always becomes the relation

$$r(\tau) = r(0), \qquad \phi(\tau) = \phi(0), \quad 0 \leqslant \tau \leqslant 1.$$

Of course, this result is exactly what are obtained from the classical Tikhonov theory [11,18] for Cases 1, 2 and method of averaging in Case 3 [13]. The advantage of using the more general AKST theory in [2] is that it has allowed us to move from case to case under the hypothetical evolution of $r(\tau)$ in one shot. Of course in retrospect, we see that $r(\tau)$ is constant as a consequence of the above arguments. We state this relation as follows.

Theorem 3.1. *The limiting dynamics for the Kuramoto system* (17)–(19) *as* $\varepsilon \to 0$ *is always*

$$r(\tau) = \textit{constant}, \qquad \phi(\tau) = \textit{constant}, \quad \textit{on the interval } 0 \leqslant \tau \leqslant 1.$$

The meaning of Theorem 3.1 is self-evident. Since $t = \frac{\tau}{\varepsilon}$, Theorem 3.1 says that as we look at our original unscaled system on t intervals $[0, \frac{1}{\varepsilon}]$ and map the graph of $r(t)$, $\phi(t)$ onto the fixed rescaled τ interval $[0, 1]$, the graphs of r and ϕ will be constant. So we now see the exact meaning of Kuramoto's assumption that the order parameters r and ϕ are constant. Specifically, the order parameters r and ϕ are indeed constant as functions of τ in the limit as $\varepsilon \to 0$ of the rescaled system (17)–(19). Hence an observer at time t, say on the interval $[0, \frac{1}{\varepsilon}]$ will see r and ϕ apparently constants as for any initial data $r(0) > 0$ if the observers move further and further away from the graphs as $\varepsilon \to 0+$. But there is a subtle proviso: our convergence to the limit nominal slow system yielding r and ϕ constants is weak-$*$ $L_\infty([0, 1])$. Hence convergence is much weaker than say the uniform convergence of a sequence of continuous functions and does not give pointwise information. Hence for example an initial layer may appear and not be recognized by our rather weak convergence.

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